Free Boolean extensions of Heyting algebras

Michał Stronkowski

Warsaw University of Technology

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closure and Heyting algebras

closure algebras = modal algebras satisfying $p \geqslant \Box p = \Box\Box p$

$\mathbf{M}$ - closure algebas
$\mathrm{O}(\mathbf{M}) = \{\Box \mathrm{p} \mid \mathrm{p} \in \mathrm{M}\}$ - Heyting algebras of open elements of $\mathbf{M}$

Theorem (McKinsey, Tarski '46)

For a Heyting algebra $\mathbf{H}$ there exists a closure algebra $\mathrm{B}(\mathbf{H})$ s. t.

- $\mathrm{OB}(\mathbf{H}) = \mathbf{H}$;
- if $\mathbf{H} \leqslant \mathrm{O}(\mathbf{M})$, then $\mathrm{B}(\mathbf{H}) \cong \langle H \rangle_{\mathbf{M}}$

$\mathrm{B}(\mathbf{H})$ is called a *free Boolean extension* of $\mathbf{H}$

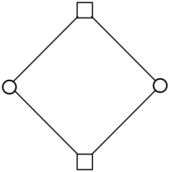
free boolean extensions

Corollary

A closure algebra is a free Boolean extension iff it is generated by its open elements

Example

A simple closure algebra $\mathbf{S}_2$ is not a free Boolean extension



$\mathbf{S}_2$

Example

$\mathbf{P}$ - closure algebra with the Boolean reduct $\mathcal{P}(\mathbb{N})$ and initial segments of $\mathbb{N}$ as its open elements. $\mathbf{P}$ is not a free Boolean extension

stable homomorphisms

M, **N** - closure algebra
$f : N \to M$ is a stable homomorphism if

- it is a Boolean homomorphism
- $\forall a \in N \quad f(\Box a) \leqslant \Box f(a)$.

Example

P admits a stable homomorphism onto $\mathbf{S}_2$ but does not admit a modal homomorphism onto $\mathbf{S}_2$

result

Theorem (Esakia '79)

A closure algebra is a free Boolean extension of a Heyting algebra iff it does not admit a stable homomorphism onto $\mathbf{S}_2$.

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Result

Easy proof (without topology)

relevance: Blok-Esakia theorem '76

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$\mathrm{Ext}\,\mathbf{Int} \cong \mathrm{NExt}\,\mathbf{Grz}$, i.e., there mappings

$$\rho\colon \mathsf{L_V}(\mathcal{G}rz) \to \mathsf{L_V}(\mathcal{H}ey); \quad \mathcal{V} \mapsto \{\mathsf{O}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{V}\}$$
$$\sigma\colon \mathsf{L_V}(\mathcal{H}ey) \to \mathsf{L_V}(\mathcal{G}rz); \quad \mathcal{Y} \mapsto \mathsf{HSP}\{\mathsf{B}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{Y}\}$$

are mutually inverse lattice isomorphisms

$\mathcal{H}ey$ - variety of all Heyting algebras
$\mathsf{L_V}(\mathcal{H}ey)$ - lattice of its subvarieties

$\mathcal{G}rz$ - variety of all Grzegorczyk algebras
$\mathsf{L_V}(\mathcal{G}rz)$ - lattice of its subvarieties

ingredients of the proof

$\mathbf{S}_2$ lemma

M - closure algebra, $f\colon M \to S_2$ - Boolean homomorphism
f is a stable homomorphism into $\mathbf{S}_2$ iff $\forall a \in M \;\; f(\Box a) \in \{0, 1\}$

kite lemma (Dwinger, Yaqub, Makinson '63)

A, **B** - Boolean algebras, **B** a proper subalgebra of **A**
There exist ultrafilters U_1, U_2 of **A** s.t.

- $U_1 \neq U_2$
- $U_1 \cap B = U_2 \cap B$

proof: $\mathbf{M}$ is a fBe $\Leftrightarrow$ $\mathbf{M} \not\twoheadrightarrow_{stab} \mathbf{S}_2$

$\Leftarrow$ $\mathbf{A}$ - Boolean reduct of $\mathbf{M}$

$\mathbf{B}$ Boolean algebra generated by O(M), $\mathbf{B} < \mathbf{A}$

take U_1, U_2 from the kite lemma

$\mathbf{S}$: with the Boolean reduct $\mathbf{A}/\mathbf{U_1} \times \mathbf{A}/\mathbf{U_1}$ and $0, 1$ open

($\mathbf{S} \cong \mathbf{S}_2$)

$a \mapsto (a/U_1, a/U_2)$ is a surjective stable homomorphism onto $\mathbf{S}$

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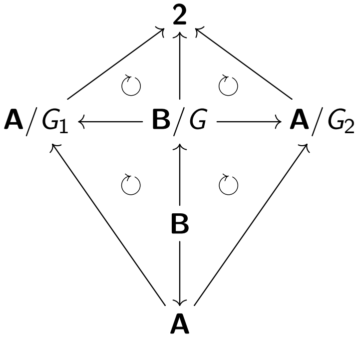
$\Rightarrow$ $\mathbf{M}$ - generated by open elements
$f : \mathbf{M} \to \mathbf{S}_2$ - stable homomorphism.
By $\mathbf{S}_2$ lemma, $f(\mathrm{O}(M)) = \{0, 1\}$
$\mathbf{M}$ is Boolean generated by $\mathrm{O}(M)$
hence $f(M) = \{0, 1\} \neq S_2$

proof of kite lemma



The end

This is all Thank you!